

Invariant manifolds and the Lax pairs for the hyperbolic type integrable equations

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Received ???, in final form ???; Published online ???

http://dx.doi.org/10.3842/SIGMA.201*.***

Abstract. In the article the problem of constructing the Lax pairs for the hyperbolic type integrable partial differential equations and their discrete counterparts is discussed. We linearize the given equation around its arbitrary solution and then look for an invariant manifold for the linearized equation. We find an invariant manifold of possibly less order containing a nontrivial dependence on constant (spectral) parameter. Usually such an invariant manifold is given by a quadratic form. For the equation found in [8] the Lax pair is constructed. Adaptation of the invariant manifolds method to the discrete versions of the hyperbolic equations is briefly discussed.

Key words: integrability, Lax pair, invariant manifold, conservation law, asymptotic diagonalization, discrete KdV equation

2010 Mathematics Subject Classification: 35Q53; 37K40

1 Introduction

In the article we constructed the Lax pair for the integrable hyperbolic type equation found in [11]

$$u_{xy} = \sqrt{1 + u_x^2} \sin u \quad (1)$$

by using the algorithm proposed in [3]. The algorithm is based on the notion of the generalized invariant manifold to the linearized equation. In order to approve that the found Lax pair is not fake we constructed a formal asymptotic representation for the Lax eigenfunction and derived from this representation an infinite series of the local conservation laws. It is commonly accepted that it is not possible to generate conservation laws from the fake Lax pairs. Discussion on the methods of constructing the Lax pairs can be found in the literature (see [1], [6], [7], [10], [12], [15], [16], [17], [18]).

Let us briefly discuss the contents of the paper. In section 2 we recall the necessary definitions. In §3 we present linear and nonlinear invariant manifolds for the linearization of (1). We give also a detailed explanation of the deriving procedure for nonlinear invariant manifold (17) playing the principal role in our algorithm. It is remarkable that the three equations: linearized equation (14), equation (17) determining the invariant manifold and its differential consequence (18) define a nonlinear Lax triad for the equation (1). The next step is to reduce the nonlinear triad to a system of linear differential equations. It is done by applying an appropriate change of the variables, defined by a pair of the quadratic forms (24), (25). The formal asymptotics for the solutions of the system (28), (29) around the singular point $\lambda = 0$ is constructed in §4. By

applying the method of the formal diagonalization (see [2], [13], [14]) the infinite series of the local conservation laws are given in an explicit form. In section 5 the invariant manifolds of the quad equations are discussed. In §6 the method of invariant manifolds for constructing the Lax pairs is adopted to the quad equations. Its application is illustrated by an example.

2 Invariant manifold and the Lax pair for the hyperbolic type integrable equations

In this section we discuss some auxiliary objects concerned to the hyperbolic type differential equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y). \quad (2)$$

Let us first recall some important definitions. Concentrate on the equation

$$g(x, y, u_k, u_{k-1}, \dots, u, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m) = 0 \quad (3)$$

defining a surface in the space of the dynamical variables $u, u_1, \bar{u}_1, u_2, \bar{u}_2, \dots$ of the equation (2), where the notations $u_j = \frac{\partial^j u}{\partial x^j}$, $\bar{u}_j = \frac{\partial^j u}{\partial y^j}$ are used. We consider differential consequences:

$$g_1(x, y, u_{k+1}, u_k, \dots, u, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m) = 0, \quad (4)$$

$$g_2(x, y, u_k, u_{k-1}, \dots, u, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m, \bar{u}_{m+1}) = 0 \quad (5)$$

of the equation (3) obtained by applying the operators of the total derivative D_x and D_y with respect to x and respectively y : $g_1 = D_x g$ and $g_2 = D_y g$ and by excluding all of the mixed derivatives of the function u due to the equation (2).

Definition 1. The surface (3) is called an invariant manifold of the equation (2) if the following equation holds

$$D_x D_y g|_{(2)-(5)} = 0. \quad (6)$$

Assume that neither of the functions g_1, g_2 vanishes identically. Then evidently the invariant surface is of the finite dimension.

In what follows we will use the linearization of the equation (2) around its arbitrary solution $u = u(x, y)$:

$$U_{xy} = aU_x + bU_y + cU, \quad (7)$$

where $a = \frac{\partial f}{\partial u_x}$, $b = \frac{\partial f}{\partial u_y}$, $c = \frac{\partial f}{\partial u}$.

Let us consider a surface defined by the equation

$$G(U_k, U_{k-1}, \dots, U, \bar{U}_1, \dots, \bar{U}_k; x, y, u, u_1, \bar{u}_1, \dots) = 0 \quad (8)$$

where $U_s = \frac{\partial^s}{\partial x^s} U$, $\bar{U}_s = \frac{\partial^s}{\partial y^s} U$ are the dynamical variables of the equation (7). Here the dynamical variables of the equation (2) u, u_1, \bar{u}_1, \dots are considered as the parameters. Find the differential consequences of the equation (8)

$$G_1(U_{k+1}, U_k, \dots, U, \bar{U}_1, \dots, \bar{U}_k; x, y, u, u_1, \bar{u}_1, \dots) = 0 \quad (9)$$

and

$$G_2(U_k, U_{k-1}, \dots, U, \bar{U}_1, \dots, \bar{U}_{k+1}; x, y, u, u_1, \bar{u}_1, \dots) = 0 \quad (10)$$

obtained by applying the operators D_x, D_y to the function G such that $G_1 = D_x G$, $G_2 = D_y G$ and then excluding all of the mixed derivatives of u and U by means of the equations (2) and (7). Everywhere below we consider invariant manifolds of the form (8) for the linearized equation (7) satisfying the condition

$$D_x D_y G|_{(2),(7)-(10)} = 0 \quad (11)$$

identically for all values of the variables $x, y, u, u_1, \bar{u}_1, \dots$.

Our main idea is to apply the reasonings above in a converse way. We examine the question whether the three equations (7)-(9) constitute the Lax triad for the equation (2)? More precisely we expect that the following consistency condition

$$D_y G_1|_{(7),(9)} = 0 \quad (12)$$

recovers equation (2). Below in the section 3 we show that such an approach is meaningful and can be used to construct the Lax pairs for integrable equations of the form (2).

3 Application of the scheme for finding the Lax pair

Let us apply the scheme of constructing the Lax pair mentioned in the end of the previous section to the following integrable hyperbolic type equation found in [11]:

$$u_{xy} = \sqrt{1 + u_x^2} \sin u. \quad (13)$$

First we look for the invariant manifolds for the linearized equation

$$U_{xy} = \sqrt{1 + u_x^2} \cos u U + \frac{u_x \sin u}{\sqrt{1 + u_x^2}} U_x. \quad (14)$$

In the paper [3] the following proposition is proved.

Proposition 1. *Equation:*

$$U_{yy} - u_y \cot u U_y + \frac{\lambda u_y}{\sqrt{1 + u_x^2} \sin u} U_x - (\lambda + \sin^2 u) U = 0 \quad (15)$$

defines an invariant manifold for the linearized equation (14). Equation of the form (9), obtained by applying D_x to (15) is as follows

$$U_{xx} - \left(\cot u + \frac{u_{xx}}{1 + u_x^2} \right) u_x U_x + \frac{u_x \sqrt{1 + u_x^2}}{\lambda \sin u} U_y - \frac{(1 + u_x^2)}{\lambda} U = 0. \quad (16)$$

It can be verified by a direct computation that the triple (14)-(16) defines a Lax triad for the equation (13), i.e. the consistency condition of these three equations leads to the equation (13). The surface defined by (15), (16) is three dimensional for any fixed function $u = u(x, y)$, since the variables U, U_x, U_y are the free variables. Therefore the triple (14)-(16) produces a Lax pair realized in 3x3 matrices (see [3]). Below we find a two-dimensional invariant surface which is nonlinear in contrast to (15), (16).

Proposition 2. *Equation of the form*

$$U_y - \frac{\lambda \cos u}{\sqrt{1 + u_x^2}} U_x - \frac{\sqrt{\lambda + 1} \sin u}{\sqrt{1 + u_x^2}} \sqrt{(1 + u_x^2) U^2 - \lambda U_x^2} = 0 \quad (17)$$

defines an invariant manifold for the equation (14). The corresponding equation (9) takes the form

$$U_{xx} - \frac{u_x u_{xx}}{1 + u_x^2} U_x - \frac{1 + u_x^2}{\lambda} U + \frac{\sqrt{\lambda + 1}}{\lambda} u_x \sqrt{(1 + u_x^2) U^2 - \lambda U_x^2} = 0. \quad (18)$$

Proposition 2 is easily proved by checking the consistency condition of the equations (14), (17), (18).

Note that the invariant manifolds for linearized equations are effectively found by direct computations. The matter is that the consistency condition (11) provides an overdetermined system of the differential equations for defining the function G . The derivation of the manifold (15) is discussed in details in [3].

Now discuss how was the nonlinear invariant manifold (17) obtained. It was deduced from the known linear invariant manifold (15) by imposing an additional constraint decreasing its order. Actually we look for a constraint of the form

$$U_y = F(U, U_x, u, u_x) \quad (19)$$

consistent with the equation (15) for all values of the dynamical variables u, u_x, u_y, \dots . Then evidently function F should satisfy the equation

$$D_y(F) - U_{yy}|_{(15),(19)} = 0 \quad (20)$$

which is rewritten in the following enlarged form:

$$\begin{aligned} & -\frac{u_y}{\sin u} \left(F(U, U_x, u, u_x) \cos u \sqrt{1 + u_x^2} - F_u(U, U_x, u, u_x) \sin u \sqrt{1 + u_x^2} - \lambda U_x \right) \\ & + F_{U_x}(U, U_x, u, u_x) \cos u U (1 + u_x^2) + F_{U_x}(U, U_x, u, u_x) u_x \sin u U_x \\ & + F(U, U_x, u, u_x) F_U(U, U_x, u, u_x) \sqrt{1 + u_x^2} + F_{u_x}(U, U_x, u, u_x) \sin u (1 + u_x^2) \\ & - (\sin^2 u + \lambda) \sqrt{1 + u_x^2} U = 0. \end{aligned} \quad (21)$$

Comparison of the coefficients at the independent variable u_y in (21) yields an ordinary differential equation for F :

$$F(U, U_x, u, u_x) \cos u \sqrt{1 + u_x^2} - F_u(U, U_x, u, u_x) \sin u \sqrt{1 + u_x^2} - \lambda U_x = 0$$

which is easily solved

$$F(U, U_x, u, u_x) = \frac{\lambda \cos u}{\sqrt{1 + u_x^2}} U_x + \sin u F_1(U, U_x, u_x).$$

By substituting the obtained specification of F into (21) we get an equation which splits down into the following two equations

$$\begin{aligned} 1. \quad & \frac{\sin u}{\cos u} \left((F_1(U, U_x, u_x) F_{1,U}(U, U_x, u_x) - U(\lambda + 1))(u_x^2 + 1)^2 \right. \\ & \left. + F_{1,u_x}(U, U_x, u_x)(1 + u_x^2)^{5/2} + u_x U_x F_{1,U_x}(U, U_x, u_x)(1 + u_x^2)^{3/2} \right) = 0, \\ 2. \quad & (1 + u_x^2)^{3/2} (F_{1,U_x}(U, U_x, u_x) U (1 + u_x^2) + \lambda F_{1,U}(U, U_x, u_x) U_x) = 0. \end{aligned} \quad (22)$$

The latter implies

$$F_1(U, U_x, u_x) = F_2 \left(u_x, \frac{(1 + u_x^2)U^2 - \lambda U_x^2}{1 + u_x^2} \right).$$

We replace F_1 in the first equation in (22) due to the obtained formula where we use notation $\theta = U^2 - \frac{\lambda}{1 + u_x^2} U_x^2$. As a result we get

$$(2F_{2,\theta}(u_x, \theta)F_2(u_x, \theta) - \lambda - 1)U + \sqrt{1 + u_x^2}F_{2,u_x}(u_x, \theta) = 0.$$

Since U is an independent variable we have two equations which give $F_2(u_x, \theta) = F_2(\theta)$ and $F_2(\theta) = \sqrt{(\lambda + 1)\theta + c}$. Let us set $c = 0$. Now we are ready to write down the final form of the searched function F :

$$U_y = \frac{\lambda \cos u}{\sqrt{1 + u_x^2}} U_x + \frac{\sqrt{\lambda + 1} \sin u}{\sqrt{1 + u_x^2}} \sqrt{(1 + u_x^2)U^2 - \lambda U_x^2} \quad (23)$$

which coincides with (17). Evidently under the constraint (23) equation (15) turns into (18).

Let us construct now a linear Lax pair for the equation (13) by using equations (17), (18), (14). To this end we introduce new variables φ, ψ instead of U, U_x by using the following quadratic forms

$$U = \varphi^2 + \psi^2, \quad (24)$$

$$U_x = \frac{2}{\sqrt{\lambda}} \sqrt{1 + u_x^2} \varphi \psi. \quad (25)$$

The consistency condition of (24), (25) gives rise to an equation

$$\varphi_x \varphi + \psi_x \psi - \frac{1}{\sqrt{\lambda}} \sqrt{1 + u_x^2} \varphi \psi = 0. \quad (26)$$

Similary the consistency of (25) and (23) yields

$$\varphi_x \psi + \psi_x \varphi + \frac{1}{2\sqrt{\lambda}} \left(u_x \sqrt{\lambda + 1} \sqrt{(\varphi - \psi)^2 (\varphi + \psi)^2} - \sqrt{u_x^2 + 1} (\varphi^2 + \psi^2) \right) = 0. \quad (27)$$

Surprisingly the system of the equations (26), (27) turned out to be linear

$$\begin{cases} \varphi_x = \frac{1}{2\sqrt{\lambda}} \left(\sqrt{1 + u_x^2} - \sqrt{\lambda + 1} u_x \right) \psi, \\ \psi_x = \frac{1}{2\sqrt{\lambda}} \left(\sqrt{1 + u_x^2} + \sqrt{\lambda + 1} u_x \right) \varphi. \end{cases} \quad (28)$$

It defines the x -part of the Lax pair. In order to obtain the y -part we apply the operator D_y to both sides of the equations (24), (25) and simplify due to the equations (14), (17). As a result we again get a linear system

$$\begin{cases} \varphi_y = -\frac{1}{2} \sqrt{\lambda + 1} \sin u \varphi + \frac{1}{2} \sqrt{\lambda} \cos u \psi, \\ \psi_y = \frac{1}{2} \sqrt{\lambda} \cos u \varphi + \frac{1}{2} \sqrt{\lambda + 1} \sin u \psi \end{cases} \quad (29)$$

which defines the y -part of the pair. Equations (28), (29) constitute a Lax pair for the equation (13).

4 Formal diagonalization of the found Lax pairs and the local conservation laws

In this section we reduce the Lax pair to a formally diagonal form by applying the method developed in [2], [13], [14].

By the linear transformation $\Phi = \tilde{T}Y$ where $\Phi = (\varphi, \psi)^T$ and $\tilde{T} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ we reduce (28), (29) to the following form

$$Y_x = AY, \quad Y_y = GY. \quad (30)$$

Here the matrices A, G are given by

$$A = \begin{pmatrix} \frac{1}{2\sqrt{\lambda}}\sqrt{1+u_x^2} & -\frac{\sqrt{\lambda+1}}{2\sqrt{\lambda}}u_x \\ \frac{\sqrt{\lambda+1}}{2\sqrt{\lambda}}u_x & -\frac{1}{2\sqrt{\lambda}}\sqrt{1+u_x^2} \end{pmatrix}, \quad G = \begin{pmatrix} \frac{\sqrt{\lambda}}{2}\cos u & \frac{\sqrt{\lambda+1}}{2}\sin u \\ \frac{\sqrt{\lambda+1}}{2}\sin u & -\frac{\sqrt{\lambda}}{2}\cos u \end{pmatrix}.$$

The matrices A, G have singularities at the points $\lambda = 0$, $\lambda = -1$ and $\lambda = \infty$. Let us expand A around $\lambda = 0$:

$$A = A_{-1}\lambda^{-1/2} + A_1\lambda^{1/2} + A_3\lambda^{3/2} + A_5\lambda^{5/2} + \dots \quad (31)$$

According to the general theory (see [2], [13], [14]) we look for a formal change of the variables $Y = T\Psi$ transforming (30) to the form

$$\Psi_x = h\Psi, \quad \Psi_y = S\Psi, \quad (32)$$

where T, h and S are formal power series:

$$T = \sum_{j=0}^{\infty} T_j \lambda^{j/2}, \quad h = \sum_{j=-1}^{\infty} h_j \lambda^{j/2}, \quad S = \sum_{j=0}^{\infty} S_j \lambda^{j/2}. \quad (33)$$

The matrices h_j are assumed to be diagonal. Assuming that for $\forall i \geq 1$ all diagonal entries of T_i vanish we find the coefficients of the series T and h from the equation $T_x = AT - Th$. By comparing the coefficients in

$$\sum_{j=0}^{\infty} D_x(T_j) \lambda^{j/2} = (A_{-1}\lambda^{-1/2} + A_1\lambda^{1/2} + \dots) \sum_{j=0}^{\infty} T_j \lambda^{j/2} - \sum_{j=0}^{\infty} T_j \lambda^{j/2} \sum_{j=-1}^{\infty} h_j \lambda^{j/2} \quad (34)$$

we obtain a sequences of the equations for defining T_j, h_j :

$$\begin{aligned} A_{-1} &= h_{-1}, \\ D_x(T_0) &= A_{-1}T_1 - T_0h_0 - T_1h_{-1}, \\ D_x(T_1) &= A_{-1}T_2 + A_1T_0 - T_0h_1 - T_1h_0 - T_2h_{-1}, \\ &\dots \end{aligned} \quad (35)$$

The system of the equations (35) is consecutively solved. Omitting the computations we give only the answers

$$\begin{aligned} h &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \lambda^{-\frac{1}{2}} + \begin{pmatrix} -\frac{1}{2} \frac{(1+\sqrt{1+u_x^2})u_{xx}}{\sqrt{1+u_x^2}u_x} & 0 \\ 0 & -\frac{1}{2} \frac{(1+\sqrt{1+u_x^2})u_{xx}}{\sqrt{1+u_x^2}u_x} \end{pmatrix} \\ &+ \begin{pmatrix} -h_{1,11} & 0 \\ 0 & h_{1,11} \end{pmatrix} \lambda^{\frac{1}{2}} + \dots, \end{aligned}$$

where $h_{1,11} = \frac{(\sqrt{1+u_x^2}-1)u_{xxx}}{2\sqrt{1+u_x^2}u_x} + \frac{u_x^2}{4} - \frac{(\sqrt{1+u_x^2}(2+3u_x^2)-4u_x^2-2)u_{xx}^2}{4u_x^2(1+u_x^2)^{\frac{3}{2}}}$ and

$$\begin{aligned} T &= \begin{pmatrix} u_x & \sqrt{1+u_x^2}-1 \\ \sqrt{1+u_x^2}-1 & u_x \end{pmatrix} + \begin{pmatrix} 0 & \frac{(\sqrt{1+u_x^2}-1)u_{xx}}{\sqrt{1+u_x^2}u_x} \\ -\frac{(\sqrt{1+u_x^2}-1)u_{xx}}{\sqrt{1+u_x^2}u_x} & 0 \end{pmatrix} \lambda^{\frac{1}{2}} \\ &+ \begin{pmatrix} 0 & T_{2,12} \\ T_{2,12} & 0 \end{pmatrix} \lambda + \dots, \end{aligned}$$

where $T_{2,12} = -\frac{1}{2}(\sqrt{1+u_x^2} - 1) + \frac{(\sqrt{1+u_x^2}-1)u_{xxx}}{\sqrt{1+u_x^2}u_x} - \frac{(\sqrt{1+u_x^2}(2+3u_x^2)-4u_x^2-2)u_{xx}^2}{2u_x^2(1+u_x^2)^{\frac{3}{2}}} + \frac{1}{2}u_x^2$.

For known T and h the series S is defined as follows

$$S = T^{-1}GT - T^{-1}T_y = \sum_{j=0}^{\infty} S_j \lambda^{j/2} \quad (36)$$

due to the expansion of G around $\lambda = 0$: $G = G_0 + G_1 \lambda^{\frac{1}{2}} + G_2 \lambda + G_3 \lambda^2 + G_4 \lambda^3 + \dots$. It can be verified that the coefficients of the series S are diagonal:

$$S = \begin{pmatrix} -\frac{1}{2} \frac{u_x \sin u}{\sqrt{1+u_x^2}-1} & 0 \\ 0 & -\frac{1}{2} \frac{u_x \sin u}{\sqrt{1+u_x^2}-1} \end{pmatrix} + \begin{pmatrix} -\frac{\sin u(2\sqrt{1+u_x^2}-u_x^2-2)u_{xx}}{2u_x^2(\sqrt{1+u_x^2}-u_x^2-1)} + \frac{1}{2} \cos u & 0 \\ 0 & \frac{\sin u(2\sqrt{1+u_x^2}-u_x^2-2)u_{xx}}{2u_x^2(\sqrt{1+u_x^2}-u_x^2-1)} - \frac{1}{2} \cos u \end{pmatrix} \lambda^{\frac{1}{2}} + \dots$$

The consistency condition of the system (32)

$$D_y h = D_x S \quad (37)$$

shows that h and S are generating functions for the local conservation laws. Equation (37) generates infinite series of the conservation laws for the equation (13). We give two of them in an explicit form

$$\begin{aligned} 1) \quad D_y \left(\frac{u_{xx}(1+\sqrt{1+u_x^2})}{u_x \sqrt{1+u_x^2}} \right) &= D_x \left(\frac{u_x \sin u}{\sqrt{1+u_x^2}-1} \right), \\ 2) \quad D_y \left(-\frac{u_{xxx}(\sqrt{1+u_x^2}-1)}{u_x \sqrt{1+u_x^2}} - \frac{1}{2}u_x^2 + \frac{u_{xx}^2(\sqrt{1+u_x^2}(2+3u_x^2)-4u_x^2-2)}{2u_x^2(1+u_x^2)^{3/2}} \right) \\ &= D_x \left(-\frac{u_{xx} \sin u(2\sqrt{1+u_x^2}-u_x^2-2)}{u_x^2(\sqrt{1+u_x^2}-(1+u_x^2))} + \cos u \right). \end{aligned}$$

In a similar way we can investigate the Lax pair (28), (29) around the singular points $\lambda = \infty$. Here we give only two local conservation laws to the equation (13), evaluated by the same method of formal series

$$\begin{aligned} 1) \quad D_y \left(\frac{\sin u(\cos u - 2)u_x}{\cos u(\cos u - 1)} \right) &= D_x \left(\frac{(\cos u + \sin^2 u + 1)u_y}{\sin u \cos u} \right), \\ 2) \quad D_y \left(\sqrt{1+u_x^2} + \frac{(\sin^2 u + 2\cos u - 2)u_x u_y}{\sin^2 u(\cos u - 1)} \right) \\ &= D_x \left(\frac{(1 - \cos u)u_{yy}}{\sin u} + \frac{(1 - \cos u)^2 u_y^2}{2 \sin^2 u} + \frac{1}{2} \sin^2 u \right). \end{aligned}$$

5 Invariant manifolds for the quad equations

The scheme applied in the previous section can be adopted to the discrete case as well. Consider a discrete equation of the form

$$u_{n+1,m+1} = f(u_{n+1,m}, u_{n,m+1}, u_{n,m}) \quad (38)$$

defined on a quadratic graph, such that the sought function depends on two integers n and m . To any of such equation one can assign an invariant manifold by analogy with the case of hyperbolic type PDE. Below we use the standard set of the dynamical variables of the equation (38) consisting of the variables in the set $\{u_{n+i,m}\}_{-\infty}^{\infty} \cup \{u_{n,m+j}\}_{-\infty}^{\infty}$.

Let us concentrate on a surface in the space of the dynamical variables defined by the following equation

$$g(u_{n+s,m}, \dots, u_{n+1,m}, u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k}) = 0. \quad (39)$$

For the definiteness we assume the integers nonnegative and at least one of them positive. By applying the shift operators D_n, D_m acting due to the rules $D_n y(n, m) = y(n+1, m)$, $D_m y(n, m) = y(n, m+1)$ to the equation (39) we deduce two additional equations

$$g_1(u_{n+s+1,m}, \dots, u_{n+1,m}, u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k}) = 0, \quad (40)$$

$$g_2(u_{n+s,m}, \dots, u_{n+1,m}, u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k+1}) = 0, \quad (41)$$

where $g_1 = D_n g$, $g_2 = D_m g$.

Definition 2. Equation (39) defines an invariant manifold for (38) if the equality

$$D_n D_m g|_{(38)-(41)} = 0 \quad (42)$$

is satisfied.

Now we define an invariant manifold for the linearization of (38)

$$U_{n+1,m+1} = AU_{n+1,m} + BU_{n,m+1} + CU_{n,m} \quad (43)$$

where the coefficients are evaluated as follows $A = \frac{\partial f}{\partial u_{n+1,m}}$, $B = \frac{\partial f}{\partial u_{n,m+1}}$, $C = \frac{\partial f}{\partial u_{n,m}}$. The definition of the invariant manifold discussed above can also be applied to the linearized equation (43). However there is a peculiarity here since the coefficients A, B, C of the equation depend of the dynamical variables $u_{n+i,m}$, $u_{n,m+j}$ of the equation (38). Therefore the linearized equation (43) is actually a family of the equations labeled by $u_{n,m}, u_{n+1,m}, u_{n,m+1}, \dots$. We are interested in invariant manifolds which also are families of surfaces depending on $u_{n,m}$ and its shifts:

$$G(U_{n+s,m}, \dots, U_{n+1,m}, U_{n,m}, U_{n,m+1}, \dots, U_{n,m+k}; u_{n,m}, u_{n+1,m}, \dots) = 0. \quad (44)$$

By applying the shift operators and excluding the mixed shifts $u_{n+i,m+j}$, $U_{n+i,m+j}$ on virtue of the equations (38) and (42) respectively, we obtain equations

$$G_1(U_{n+s+1,m}, \dots, U_{n+1,m}, U_{n,m}, U_{n,m+1}, \dots, U_{n,m+k}; u_{n,m}, u_{n+1,m}, \dots) = 0, \quad (45)$$

$$G_2(U_{n+s,m}, \dots, U_{n+1,m}, U_{n,m}, U_{n,m+1}, \dots, U_{n,m+k+1}; u_{n,m}, u_{n+1,m}, \dots) = 0, \quad (46)$$

where $G_1 = D_n G$, $G_2 = D_m G$. According to the definition the following condition

$$D_n D_m G|_{(38),(43);(44)-(46)} = 0 \quad (47)$$

should hold identically.

We conjecture that the invariant manifold for the linearized equation (43) can effectively be used for constructing the Lax pair to the equation (38). Note that for the integrable models of the form (38) satisfying the consistency around a cube condition the algorithms of constructing the Lax pairs have been proposed in [1], [10] (see also [15]).

6 An example of evaluating the Lax pair for the quad equation via invariant manifold

Let us illustrate the application of the method of invariant manifolds for constructing the Lax pair in the discrete case with an example. As a touchstone we take the well known discrete version of the KdV equation (see [5], [9]):

$$q_{n+1,m+1} = q_{n,m} + c \left(\frac{1}{q_{n+1,m}} - \frac{1}{q_{n,m+1}} \right). \quad (48)$$

Below we use the abbreviated notation as follows. We put $q_{i,j}$ instead of $q_{n+i,m+j}$ to rewrite (48) as $q_{11} = q + \frac{c}{q_{10}} - \frac{c}{q_{01}}$. Then the linearization of (48) found due to (43) takes the form

$$Q_{11} = Q - c \left(\frac{1}{q_{10}^2} Q_{10} - \frac{1}{q_{01}^2} Q_{01} \right). \quad (49)$$

A linear substitution $Q = U_{10} - U_{01}$ leads (49) to the following form

$$U_{11} = U - \frac{c}{q^2} (U_{10} - U_{01}). \quad (50)$$

For our further purpose equation (50) is more preferable since the coefficients of this equation depend only on q , while the coefficients of (49) depend on two shifts of the variable q . By using the results of the paper [4] we find a nonlinear invariant manifold for the equation (50):

Proposition 3. *An equation of the form*

$$U_{20} = U + \frac{2\sqrt{\lambda}}{p_{10}} \sqrt{UU_{10}} + \frac{\lambda}{p_{10}^2} U_{10}, \quad (51)$$

where $p_{10} = \frac{q}{qq_{10}+c}$ and λ is an arbitrary constant parameter, defines an invariant manifold for the linear equation (50).

By direct computations one can check that the surface (51) satisfies the requirements of the Definition 2.

Let us apply the operator D_m to both sides of the equation (51) and exclude the mixed shifts U_{21}, U_{11} due to the linear equation (50), then exclude the variable U_{20} due to (51). As a result we come to the equation

$$\begin{aligned} & \left(1 + \frac{cq^2\lambda}{(q - cp_{10})^2} \right) U_{01} + \frac{\lambda q^2}{(q - cp_{10})^2} (q^2 U - cU_{10}) + \frac{2\sqrt{\lambda}q}{q - cp_{10}} \sqrt{q^2 UU_{01} - c(U_{10} - U_{01})U_{01}} \\ &= \frac{c^2 p_{10}^2}{(q - cp_{10})^2} U_{01} - \frac{q(2cp_{10} + c\lambda q - q)}{(q - cp_{10})^2} U_{10} - \frac{2c\sqrt{\lambda}q^2 p_{10}}{(q - cp_{10})^2} \sqrt{UU_{10}}. \end{aligned} \quad (52)$$

Find the variable U_{01} from the equation (52) by reducing it to a quadratic equation. As a result we get

$$U_{01} = \frac{1}{1 - c\lambda} \left(U_{10} + 2\sqrt{\lambda}q\sqrt{UU_{10}} + \lambda q^2 U \right). \quad (53)$$

We can rewrite the linear equation (50) in the following form

$$U_{11} = \frac{1}{1 - c\lambda} \left(\frac{\lambda c^2}{q^2} U_{10} + \frac{2c\sqrt{\lambda}}{q} \sqrt{UU_{10}} + U \right) \quad (54)$$

on virtue of the equation (53).

Summarizing the reasonings above we conclude that equations (51), (53), (54) produce a nonlinear Lax pair to the equation (48):

$$\begin{aligned} D_n U &= U_{10}, \\ D_n U_{10} &= U + \frac{2\sqrt{\lambda}}{p_{10}} \sqrt{U U_{10}} + \frac{\lambda}{p_{10}^2} U_{10}; \end{aligned} \quad (55)$$

$$\begin{aligned} D_m U &= \frac{1}{1 - c\lambda} \left(U_{10} + 2\sqrt{\lambda}q \sqrt{U U_{10}} + \lambda q^2 U \right), \\ D_m U_{10} &= \frac{1}{1 - c\lambda} \left(\frac{\lambda c^2}{q^2} U_{10} + \frac{2c\sqrt{\lambda}}{q} \sqrt{U U_{10}} + U \right). \end{aligned} \quad (56)$$

Evidently the factor $(1 - c\lambda)^{-1}$ in (56) is removed by changing $U_{n,m} = (1 - c\lambda)^{-m} \tilde{U}_{n,m}$. Replace the variables ones more by taking $\tilde{U} = \varphi^2$, $\tilde{U}_{10} = \varphi_{10}^2$. Omitting the computations we give only the resulting linear system

$$\Psi_{10} = A\Psi, \quad (57)$$

$$\Psi_{01} = B\Psi, \quad (58)$$

where $\Psi = (\varphi_{10}, \varphi)^T$ and the matrix coefficients are

$$A = \begin{pmatrix} -\frac{\sqrt{\lambda}(qq_{10}+c)}{q} & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{c\sqrt{\lambda}}{q} & -1 \\ 1 & \sqrt{\lambda}q \end{pmatrix}. \quad (59)$$

The obtained Lax pair coincides with that found earlier (see [5], [9]).

Acknowledgements

The authors gratefully acknowledge financial support from a Russian Science Foundation grant (project 15-11-20007).

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